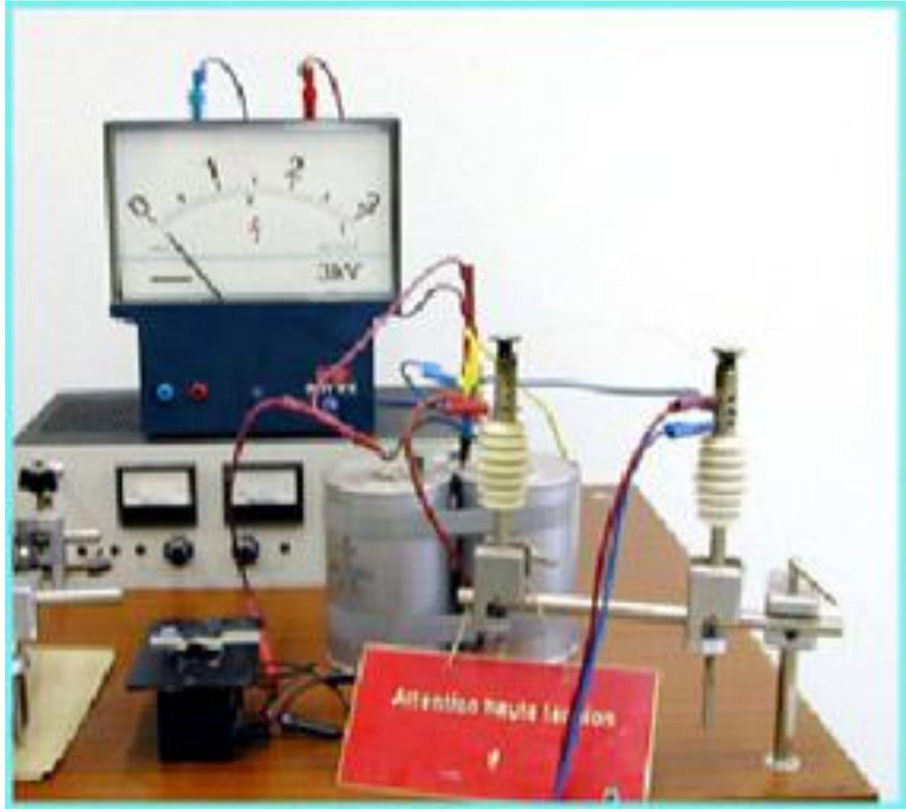
Week 7:

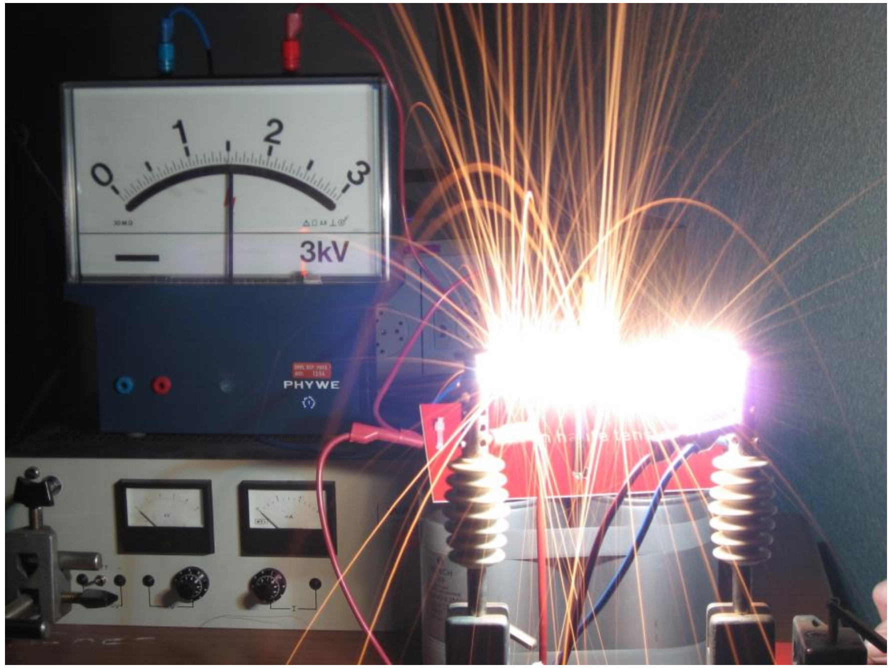
Current, Resistance, RC Circuit

Energy of a capacitor (copper wire)

DEMO

The purpose of this experiment is also to illustrate the **internal resistance** of an object (here a copper wire) by vaporizing it by **Joule effect** using a capacitor. The capacitor, storing the energy physically and not chemically, allows to provide a lot of electrical energy through the copper wire in a very short time, which will have the effect of increasing its temperature very quickly to the point of vaporizing this cable almost instantaneously.





https://auditoires-physique.epfl.ch/experiment/422/energie-dun-condensateur-fil-de-cuivre

Electric Current

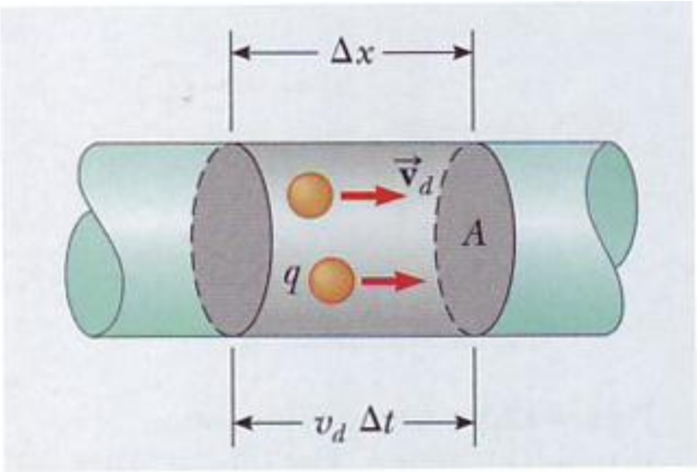
Electric field in conductors and resistivity:

- **E inside of conductors is 0 only in static case (no current)**
- Electrons in a conductor move like molecules in a gas
- **Upon applying potential difference electrons in addition drift toward +**
- **Their motion is hindered by scattering on atoms (also on impurities) – this causes electric resistivity.**
- The higher is T, the high is chaotic thermal motion – larger resistance.

Electric current: Charge flux per unit time through a surface A

$$I \triangleq \frac{dQ}{dt} = \frac{\Delta Q}{\Delta t} = \frac{\rho A \Delta x}{\Delta t} = (nqv_d)A = JA$$

Let's consider the total charge crossing a conductor cross-section each unit of time



Note 1: 1 A = 1 C/s

Electron charge: $e \cong 1.6 \times 10^{-19}$ C

⇒ a current of 1 A consists of 6×10^{18} électrons/s through the surface A

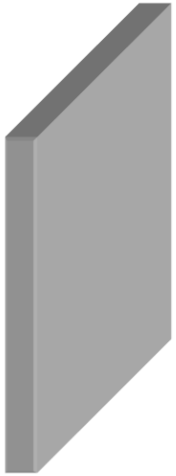
Electric Current density *J*

- current density J is **current per area** or, equivalently,

charge per area and time

unit of J: A/m^2

directions are important …

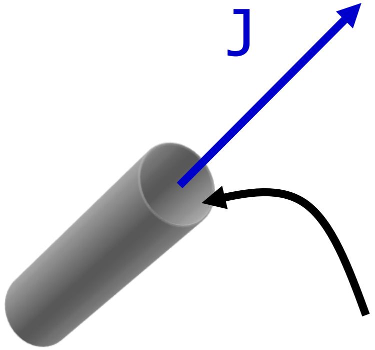


- current density $\vec{j}$ flowing through infinitesimal area $d\vec{A}$ produces infinitesimal current

$$dI = \vec{J} \cdot d\vec{A}$$

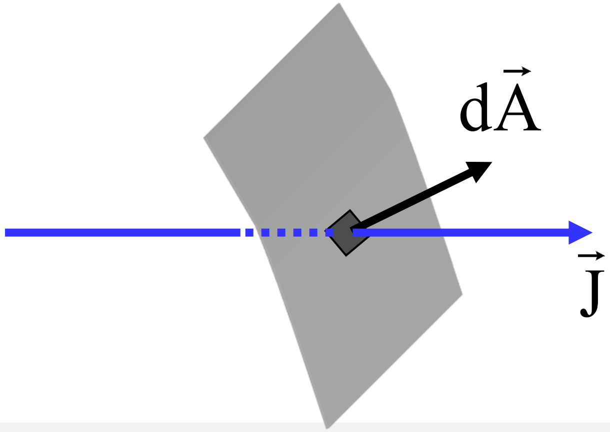
- total current passing through A is

$$I = \int_{surface} \vec{J} \cdot d\vec{A}$$



cross section A of wire

- current density is a **vector**
(direction is direction of velocity of positive charge carriers)



- if $\vec{J}$ is uniform and parallel to $d\vec{A}$:

$$I = \int_{surface} \vec{J} \cdot d\vec{A} = J \int_{surface} dA = JA \Rightarrow \boxed{J = \frac{I}{A}}$$

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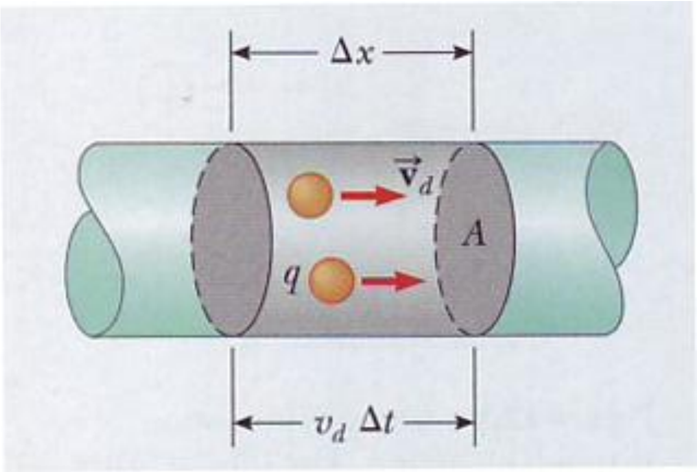


Charge density: $\rho = nq$ $[\mathrm{C/m^3}]$
Density of charged particles: n $[\mathrm{1/m^3}]$
Current density: $\mathbf{J} = nq\mathbf{v}_d = \rho\mathbf{v}_d$ $[\mathrm{A/m^2}]$
Current: $I = JA$ $[\mathrm{A}]$
Charge "drift" Speed: $\mathbf{v}_d$ $[\mathrm{m/s}]$

Note 1: $1\ \mathrm{A} = 1\ \mathrm{C/s}$
Electron charge: $e \cong 1.6 \times 10^{-19}$ C
$\Rightarrow$ a current of 1 A consists of 6×10^{18} électrons/s through the surface A

typical currents:

- 100 W light bulb: roughly 1A
- car starter motor: roughly 200A
- TV, computer, phone: nA to mA

current is a scalar (not a vector)

- has a **sign** associated with it
- **conventional** current is flow of **positive** charge

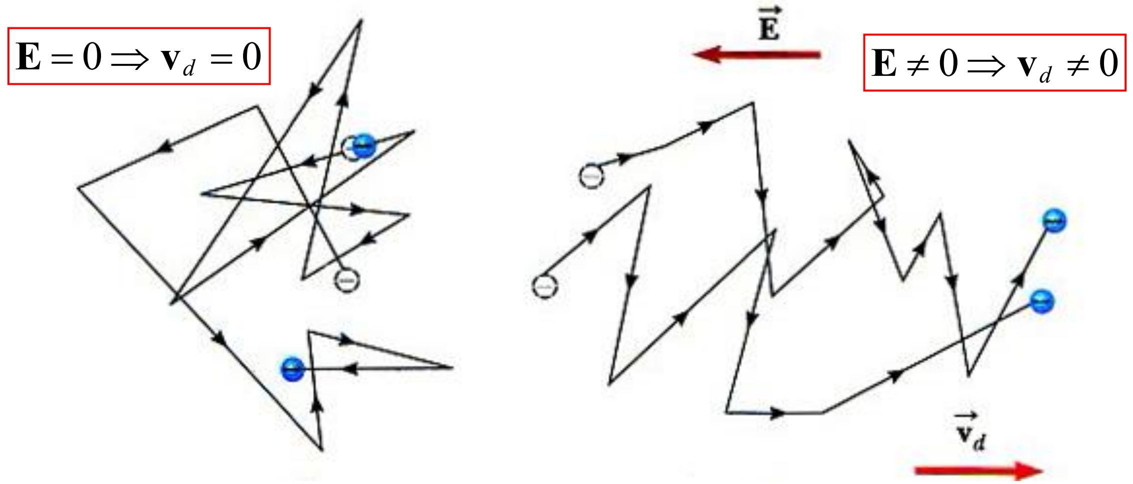
Currents in Materials

Metals are conductors because they have "free" electrons, which are not bound to metal atoms.

In a cubic meter of a typical conductor there roughly 10^{28} free electrons, moving with typical speeds of 1,000,000 m/s…

…but the electrons move in random directions, and there is no net flow of charge, until you apply an electric field.

Ohm's Law, a microscopic model



Due to collisions with fixed ions, the speed of free electrons in a solid reaches a limiting value $\mathbf{v}_d$ («drift» velocity) proportional to the electric field.
(≈ viscous friction)

For many materials (including most metals), **the ratio of the current density to the electric field is a constant s that is independent of the electric field producing the current**.

Drude Model: CURIOSITY

$$\mathbf{F} = q\mathbf{E} \quad \Rightarrow \quad \mathbf{a} = \frac{q}{m_e}\mathbf{E} \quad \Rightarrow$$

For a single charge between two collisions:

$$\mathbf{v}(t) = \mathbf{v}(0) + \frac{q}{m_e}\mathbf{E}t$$

For a large number of charges:

$$\langle \mathbf{v}(0) \rangle = 0 \Rightarrow$$

$$\langle \mathbf{v} \rangle \triangleq \mathbf{v}_d = \langle \mathbf{v}(0) \rangle + \frac{q\mathbf{E}}{m_e}\langle t \rangle \cong \frac{q}{m_e}E\tau$$

τ: average time between two collisions

$\mathbf{v}_d$: "drift" velocity (effective average speed in the direction of the field $\mathbf{E}$)

$\Longrightarrow$

$$\mathbf{v}_d = \frac{q}{m_e}\mathbf{E}\tau$$

$\Rightarrow$

$$\mathbf{J} \equiv nq\mathbf{v}_d = \frac{nq^2\mathbf{E}}{m_e}\tau = \sigma\mathbf{E}$$

$\Rightarrow$

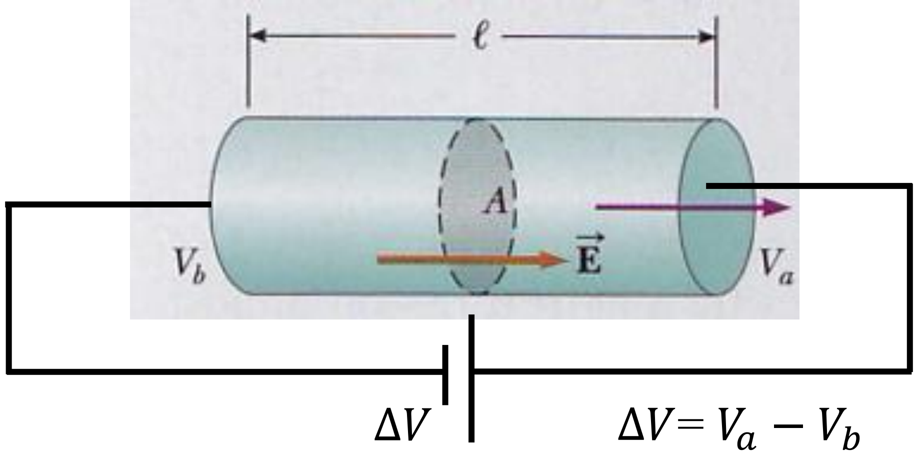
$$\mathbf{J} = \sigma\mathbf{E}$$ Ohm's law "local"

Electrical conductivity [1/Ωm=S/m]:

$$\sigma \equiv \frac{nq^2}{m_e}\tau$$

Electrical Resistivity [Ωm]: $\rho \equiv \frac{m_e}{nq^2\tau}$

Electric Current and Resistance



$\Delta V = V_a - V_b$ (generated, for instance, by a battery)

If the ends of the conducting wire are connected to a battery, all points on the loop are not at the same potential. **The battery sets up a potential difference between the ends of the loop, creating an electric field within the wire.** The electric field exerts forces on the electrons in the wire, causing them to move in the wire and therefore **creating a current**

$\mathbf{E} = -\nabla V$ stationary condition

$\Rightarrow$

$\mathbf{E}(\mathbf{x}) = E\ \hat{\mathbf{x}} = \text{cost}$

$\Rightarrow$

$\int_0^l \mathbf{E}\cdot d\mathbf{l} = \Delta V \qquad \int_0^l \mathbf{E}\cdot d\mathbf{l} = E\ l$

$\Rightarrow$

$$E = \frac{\Delta V}{l}$$

E is not zero in the conductor.
(stationary current but not static charges…
therefore the condition is not "electrostatic")

conductor	ρ [Ω m]
Copper	1.7 10^{-8}
Iron	10^{-7}
Glass	10^{+12}

In some materials, the current density is proportional to the electric field

$$\mathbf{J} = \frac{I}{A} = \boldsymbol{\sigma} E = \sigma\frac{\Delta V}{l}$$ Ohm's law "local"

(σ: conductivity)

$\Rightarrow$

$$\Delta V = \frac{l}{\sigma}\mathbf{J} = \left(\frac{l}{\sigma A}\right)I = \boldsymbol{R}I$$

R is called resistance

$\Rightarrow$

$\Delta V = RI$ Ohm's law "global"

$$R \triangleq \left(\frac{l}{\sigma A}\right) = \boldsymbol{\rho}\frac{l}{A}$$

(ρ: resistivity)

ρ is called resistivity

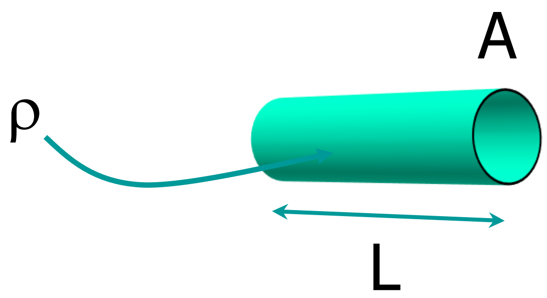
Unit of Resistance ***R***: 1 Ω = 1V / 1 A

Unit of resistivity ρ: Ω·m

Resistance of a wire: Summary

current in a wire:

- length L, cross section A
- material of resistivity ρ



- resistance of wire (or other device) measures how easily charge flows through it

start from $E = \rho J$

$$V = EL = \rho JL = \rho \frac{I}{A} L = IR$$

$$R = \frac{\rho L}{A}$$

- the longer a wire, the harder it is to push electrons through it
- the greater the cross-sectional area, the "easier" it is to push electrons through it
- the greater the resistivity, the "harder" it is for the electrons to move in the material

Ohm' law (device version)

$$V = IR$$

resistance of the wire,

unit $\frac{V}{A} = \Omega$ (Ohm)

Distinguish:
Resistivity = material's property
Resistance = device property

Notes:

1. Problem of the notation

Beware of confusion between: ρ (resistivity) and ρ (charge density in a volume),
σ (conductivity) and σ (surface charge density)

2. Speed of thermal agitation and mean time between two collisions:

$\langle \mathbf{v}(0) \rangle = 0$ but $\langle v(0) \rangle \neq 0$ (thermal agitation of electrons)

$$\langle v(0) \rangle \cong \boldsymbol{v_{th}} = \sqrt{\frac{3kT}{m_e}} \cong \mathbf{10^5\ m/s} \text{ (!!) for } T \cong 300 \text{ K}$$

The Mean Free Path λ is a little larger than the distance between the atoms d_{atoms}.

So, the time between collisions is: $\tau \sim \frac{\lambda}{v_{th}} \sim \frac{10 d_{atoms}}{v_{th}} \sim \frac{10^{-9}\text{ m}}{10^5\text{ m/s}} \sim 10^{-14}$ s

3. Drift velocity:

For a copper wire($n \cong 8.5 \times 10^{28}$ électrons/m^3)
with section A=1 mm^2 with a current I=10 A ⇒

$$J = env_d = \frac{I}{A} = 10^7 \text{ A/m}^2 \quad \Rightarrow \boldsymbol{v_d} = \frac{J}{en} \cong \mathbf{1\ mm/s} \text{ (!!)} \quad \Rightarrow \quad \boldsymbol{v_d} << \boldsymbol{v_{th}} \text{ (!!)}$$

4. Field $\mathbf{E} = 0$ or $\mathbf{E} \neq 0$ in a conductor?

$\mathbf{E} = 0$ for a conductor in electrostatic conditions (static charges, so for $\mathbf{J}$=0).

For a Perfect Conductor (with $\sigma = \infty$), $\mathbf{E} = \frac{\mathbf{J}}{\sigma} = 0$ also for $\mathbf{J} \neq 0$.

For a real conductor $\mathbf{E} = 0$ in electrostatic conditions ("static charges") but $\mathbf{E} \neq 0$ for $\mathbf{J} \neq 0$.

5. Phenomenologically, the current density in many systems obeys Ohm's law.
But Ohm's law is not a "mathematical" consequence of Maxwell's equations.

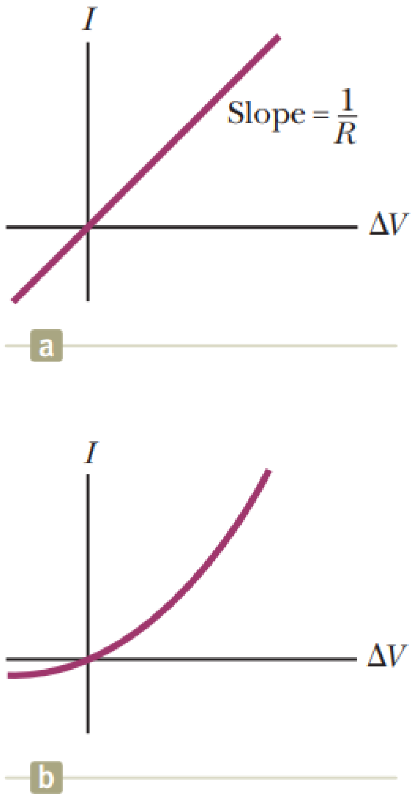


Figure 27.7 (a) The current–potential difference curve for an ohmic material. The curve is linear, and the slope is equal to the inverse of the resistance of the conductor. (b) A nonlinear current–potential difference curve for a junction diode. This device does not obey Ohm's law.

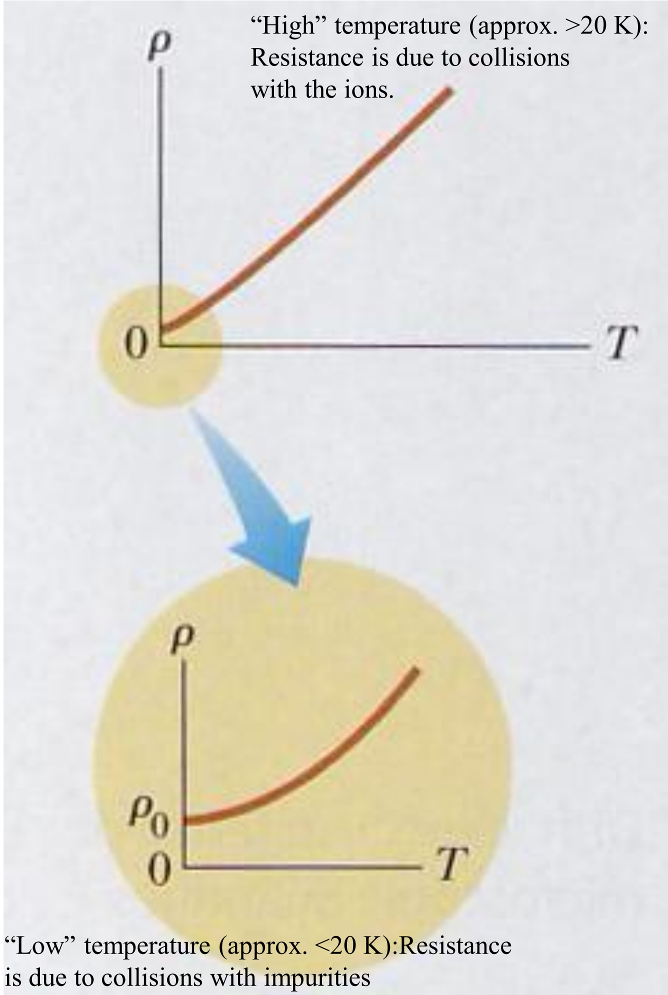
Table 27.2 Resistivities and Temperature Coefficients of Resistivity for Various Materials

Material	Resistivity[a] ($\Omega \cdot m$)	Temperature Coefficient[b] α [$(°C)^{-1}$]
Silver	1.59×10^{-8}	3.8×10^{-3}
Copper	1.7×10^{-8}	3.9×10^{-3}
Gold	2.44×10^{-8}	3.4×10^{-3}
Aluminum	2.82×10^{-8}	3.9×10^{-3}
Tungsten	5.6×10^{-8}	4.5×10^{-3}
Iron	10×10^{-8}	5.0×10^{-3}
Platinum	11×10^{-8}	3.92×10^{-3}
Lead	22×10^{-8}	3.9×10^{-3}
Nichrome[c]	1.00×10^{-6}	0.4×10^{-3}
Carbon	3.5×10^{-5}	-0.5×10^{-3}
Germanium	0.46	-48×10^{-3}
Silicon[d]	2.3×10^{3}	-75×10^{-3}
Glass	10^{10} to 10^{14}	
Hard rubber	$\sim 10^{13}$	
Sulfur	10^{15}	
Quartz (fused)	75×10^{16}	

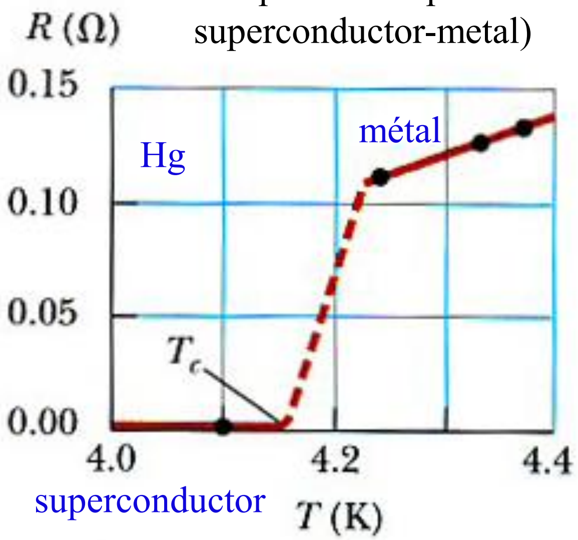
[a] All values at 20°C. All elements in this table are assumed to be free of impurities.
[b] See Section 27.4.
[c] A nickel–chromium alloy commonly used in heating elements. The resistivity of Nichrome varies with composition and ranges between 1.00×10^{-6} and $1.50 \times 10^{-6}\ \Omega \cdot m$.
[d] The resistivity of silicon is very sensitive to purity. The value can be changed by several orders of magnitude when it is doped with other atoms.

6. Resistance and Resistivity for Metals and Other Materials.

Temperature dependence of resistivity (for a metal)

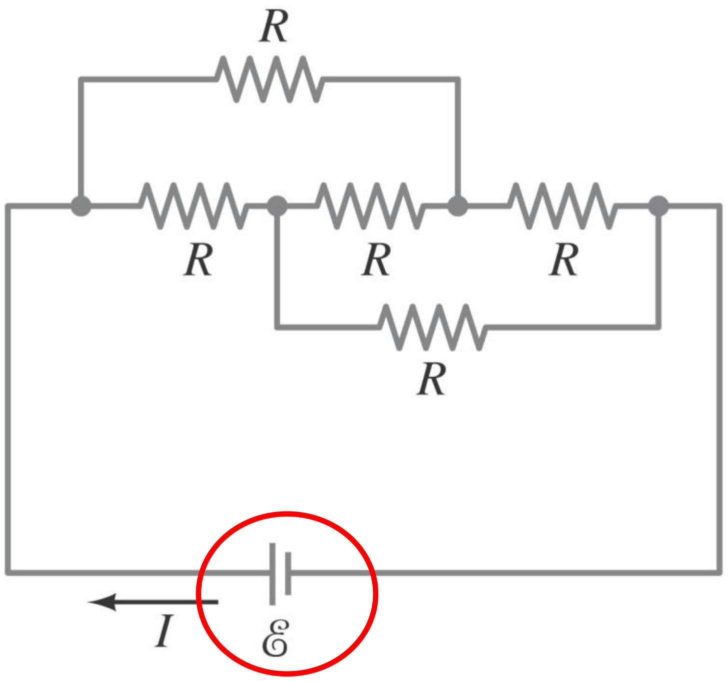


Temperature dependence of resistance(for a superconductor-metal)



One truly remarkable feature of superconductors is that once a current is set up in them, it persists without any applied potential difference (because R = 0). Steady currents have been observed to persist in superconducting loops for several years with no apparent decay!

Ideal Direct Current (DC) Circuits



$$I = \frac{V}{R}$$

Idealization (convention):

- Connecting wires have no resistance (or inductance or capacity).
- All resistance is in special elements labeled ***R***.
- All capacity is in elements labeled ***C***.

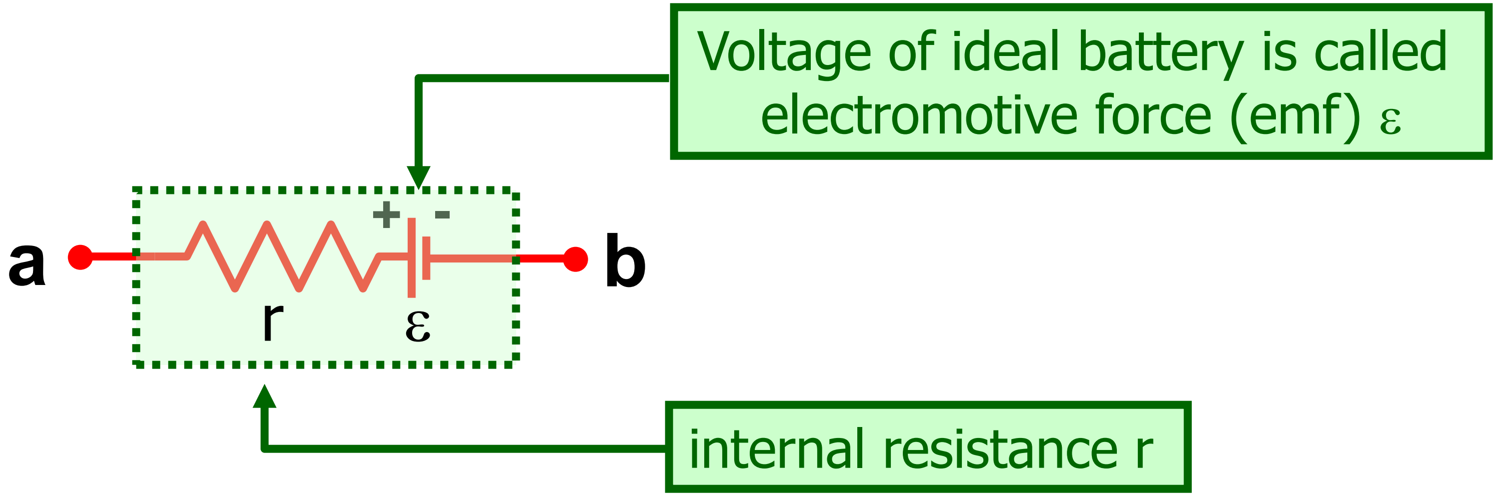
Properties of battery ?

Ideal versus real voltage sources

- **ideal** battery (or other voltage source):
  voltage does **not** depend on the current flowing
- **real** battery: voltage does depend on current,
  typically voltage **decreases** with increasing current (load)

How to model a real battery?

- real battery consists of ideal battery + internal resistance



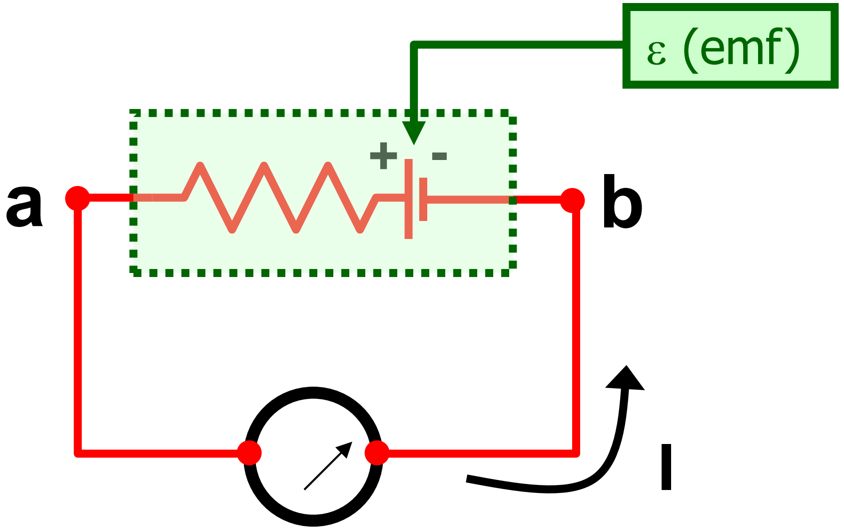
EMF and terminal voltage

The electromotive force (emf) of a voltage source is the potential difference it produces when no current is flowing.

emf is not a force!

Can the emf be measured?

- hook up a voltmeter:
- as soon as you connect the voltmeter, current flows
- you can only measure **terminal voltage** V_{ab}, but not emf ε



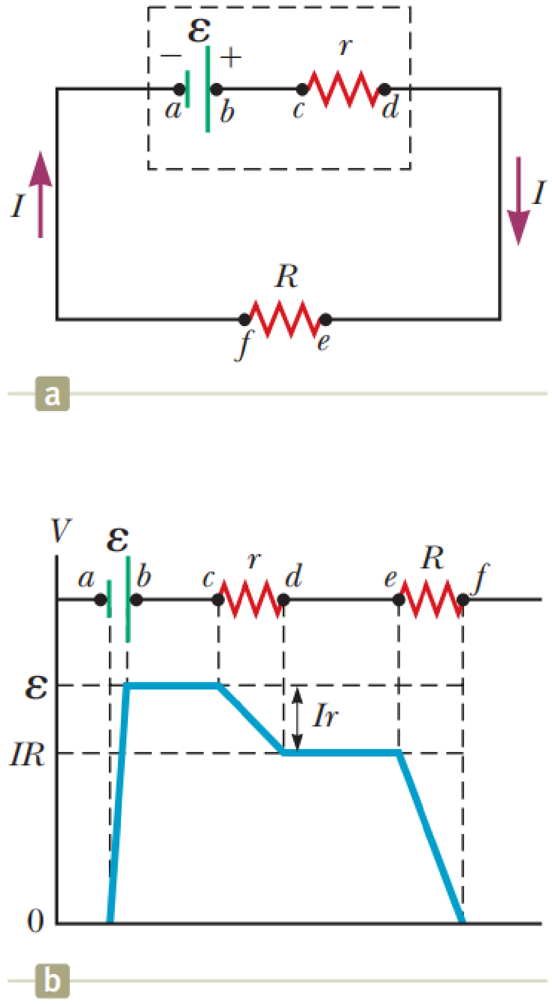
An ideal voltmeter would be able to measure ε.

The "battery" is everything inside the green box.

EMF and Terminal Voltage

We will generally use a **battery** as a **source of energy** for circuits in our discussion.

battery



Electric circuit needs battery or voltage generator to produce current – these are called, in general, sources of **electromotive force** (**emf, $\mathcal{E}$**).

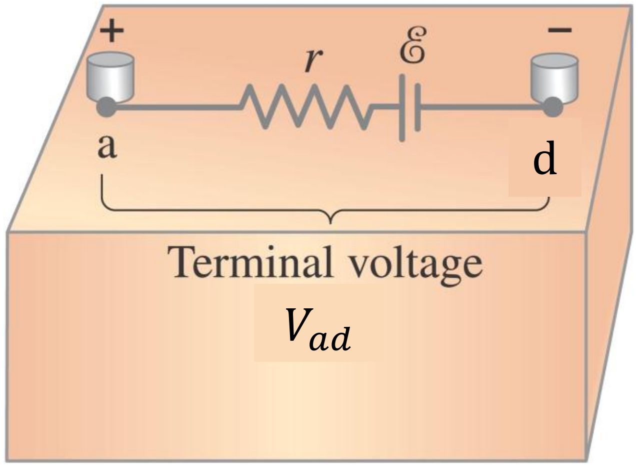
Real battery does have a small **internal resistance r**, such that:

$$V_{ad} = \mathcal{E} - Ir$$

This resistance behaves as though it were in series with the emf.

The actual voltage V_{ad} applied to an external resistor R drops upon increase of the current

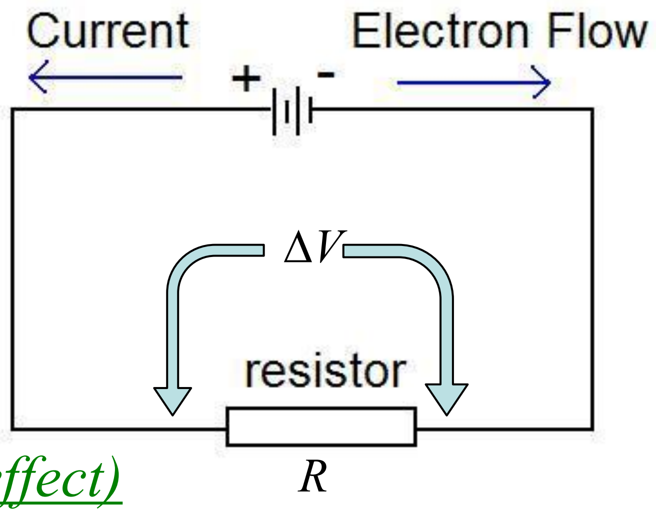
$I_{max} < \mathcal{E}/r$ ($R >> r$)



Electric Power

Consider an electric circuit with an ideal battery and a resistor :

- *The battery keeps ΔV= constant*
- *Current is from + to –*



The work spent by battery to transfer a charge $\boldsymbol{Q}$ through the resistor:

$$W = \Delta V \cdot Q$$

- *This energy is released in the resistor as heat (Joule effect)*

The **rate** at which the electric potential energy of the system decreases as the charge Q passes through the resistor:

$$P = \frac{dW}{dt} = \frac{d}{dt}(\Delta V \cdot Q) = \Delta V \cdot \frac{dQ}{dt} = \Delta V \cdot I$$

Unless noted otherwise, we shall assume the resistance of the wires is small compared with the resistance of the circuit element so that the energy delivered to the wires is negligible

POWER DISSIPATED through R

$$P = VI = V^2/R = I^2 R$$
$$(I = V/R)$$

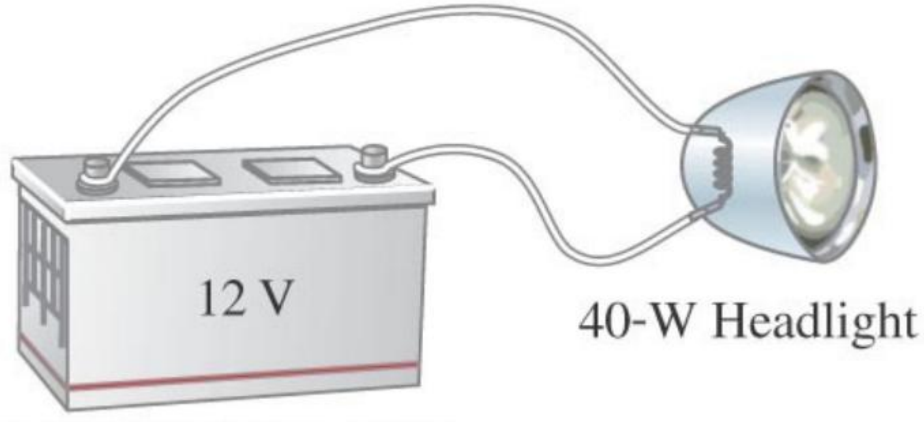
$[P] = \text{Watt} = \text{VA}$

Simple exemples for Electric Power

Example I

Example: Headlights.

Calculate the resistance of a 40-W automobile headlight designed for 12 V.



$P=V^2/R$, => $R=V^2/P = 144/40 = 3.6\ \Omega$

Example II

What you pay for on your electric bill is not power, but energy – the power consumption multiplied by the time.

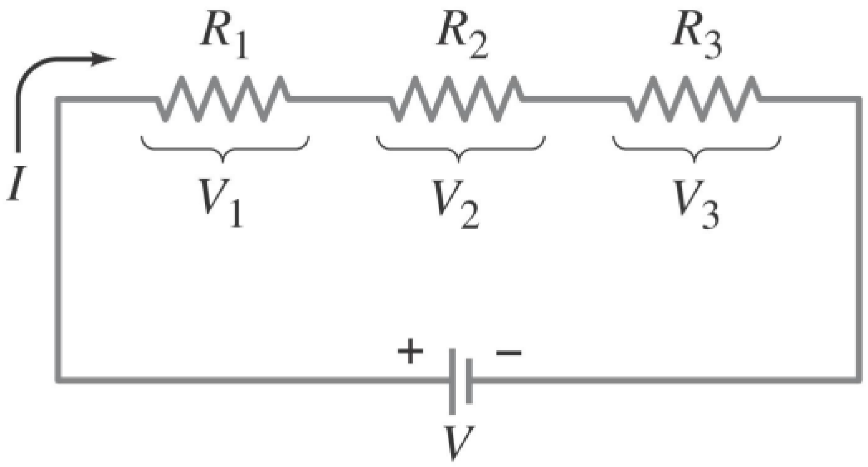
We have been measuring energy in joules, but the electric company measures it in kilowatt-hours, kWh:

$$1\ \text{kWh} = (1000\ \text{W})(3600\ \text{s}) = 3.60 \times 10^6\ \text{J}.$$

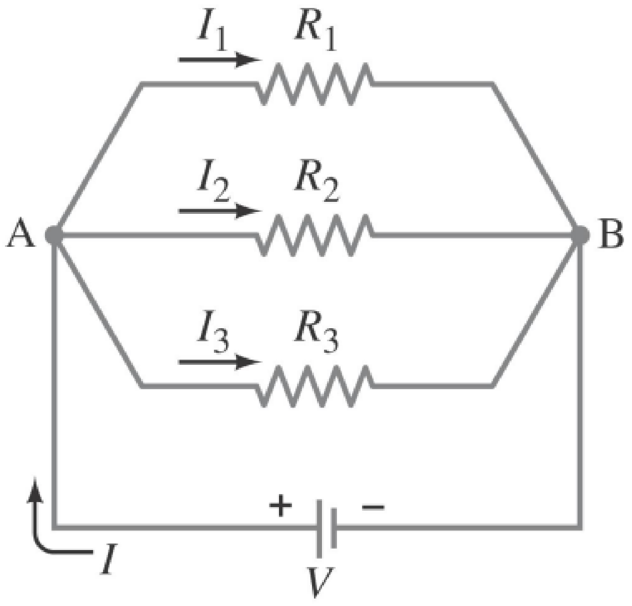
$$W = P\cdot t = V\cdot I\cdot t = V\cdot Q$$

Resistors in Series and in Parallel

Series: a single path from the battery, through each circuit element in turn, then back to the battery.



Parallel: the current is split; the voltage across each resistor is the same.

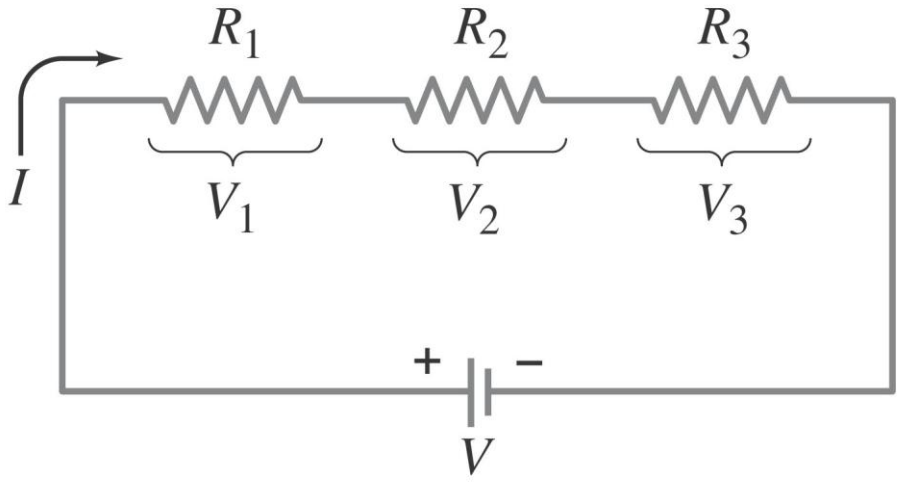


Q: What is the equivalent resistance?

The equivalent resistance has the same effect on the circuit as the series/parallel combination of resistors; that is, the equivalent resistance draws the same current I from the battery.

Resistors in Series and in Parallel

Series: single circuit; the same current.



$$V = V_1 + V_2 + V_3 = IR_1 + IR_2 + IR_3$$

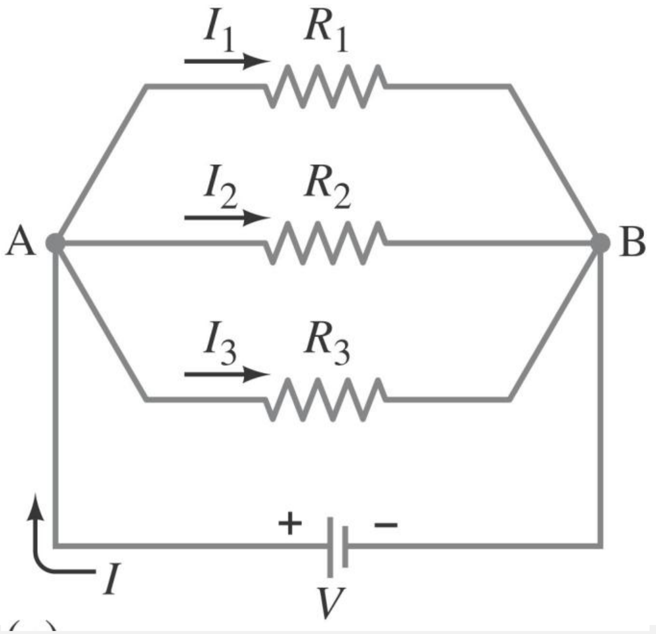
$$V = IR_{\text{series}} = IR_1 + IR_2 + IR_3$$

$$R_{\text{series}} = R_1 + R_2 + R_3$$

What is the equivalent resistance?

Parallel: the current is split; the voltage across each resistor is the same.

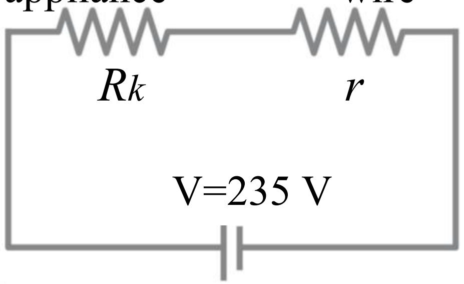
$$I = I_1 + I_2 + I_3 = \frac{V}{R_{\text{parallel}}} = \frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3} \rightarrow$$



$$\frac{1}{R_{\text{parallel}}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Electric Power at home

Let's now consider a simple everyday example. Suppose we have an electrical appliance, such as an iron or a boiler, which can be modeled as a resistor R_k.
We also take into account that real wires have a small but finite resistance, r_{wire}. We can then calculate the ratio η between the power delivered to the appliance and the total power supplied (which includes the power dissipated in the wires).



appliance

(a) $$I = \frac{V}{R_k + r_{wire}}; \quad P_k = I^2 R_k; \quad P_{wire} = I^2 r_{wire} \Longrightarrow \eta = \frac{P_k}{P_k + P_{wire}} = \frac{R_k}{R_k + r_{wire}}$$

$R_k \gg r_{wire}$ (otherwise loss of energy in the wires = heating of the wires)

(b) Power strip (***n***-sockets): ***n*** parallel connections:

$$R_n = \frac{R_k}{n} \Longrightarrow I = \frac{V}{R_k/n + r_{wire}}; \quad \eta = \frac{R_k}{R_k + n r_{wire}}$$

Too high current through a wire can heat it up and inflame (make wires short and thick; respect max current through).

burns out

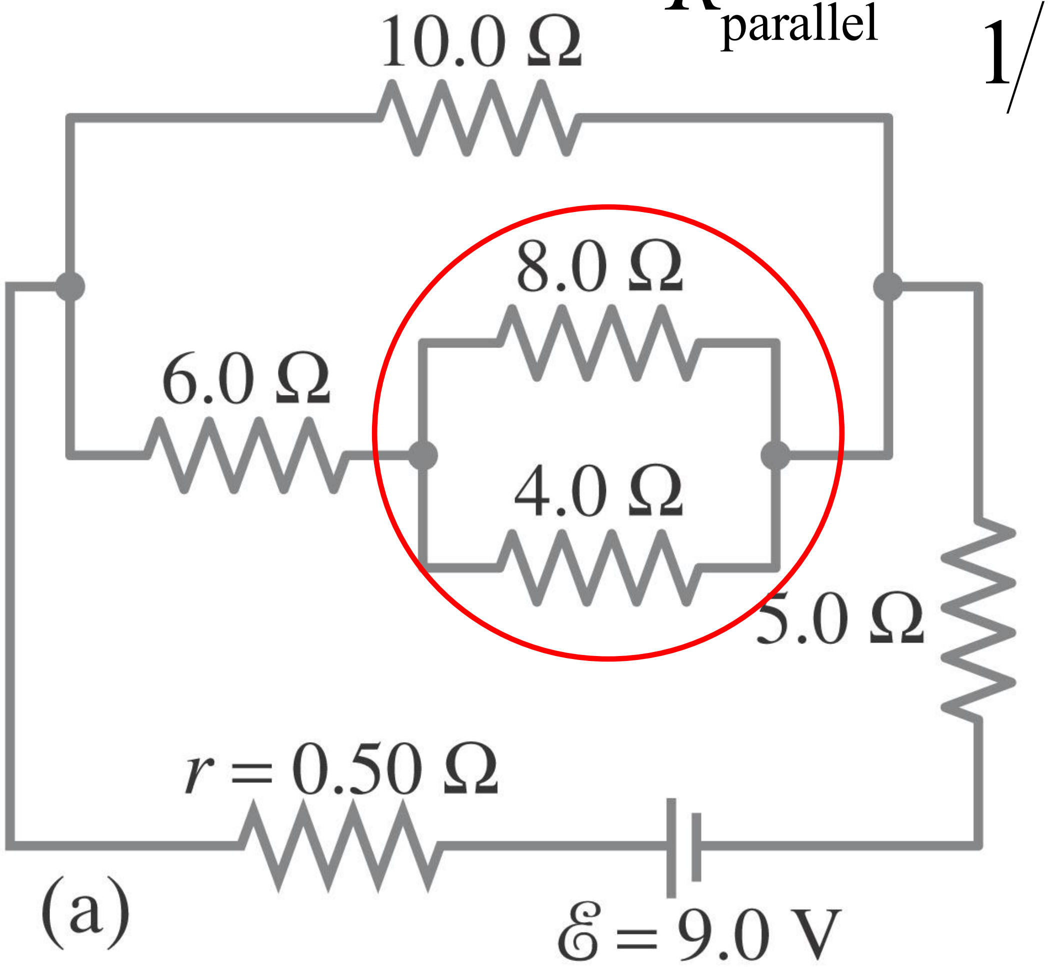
switches off

To protect respect to high currents

Resistors in Series and in Parallel

Example: Big circuit.

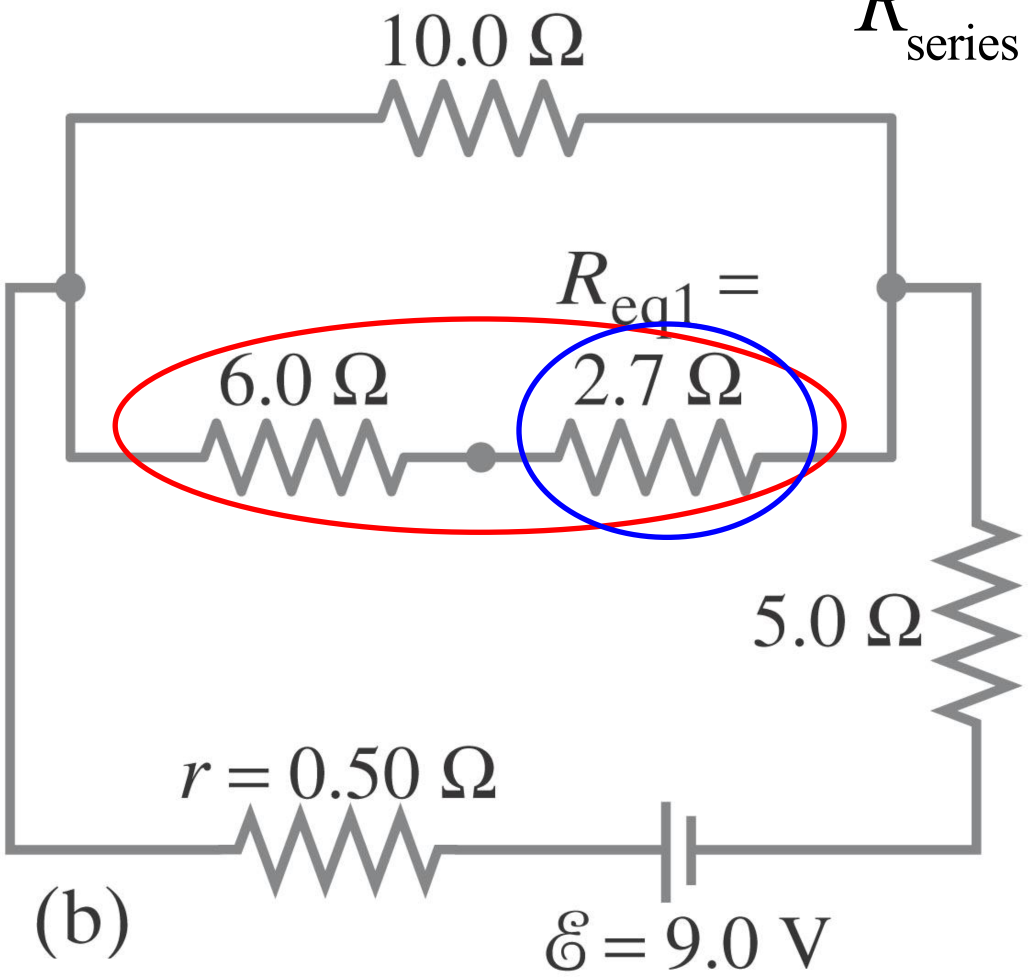
$$R_{\text{parallel}} = \frac{1}{1/R_1 + 1/R_2}$$



Resistors in Series and in Parallel

Example: Big circuit.

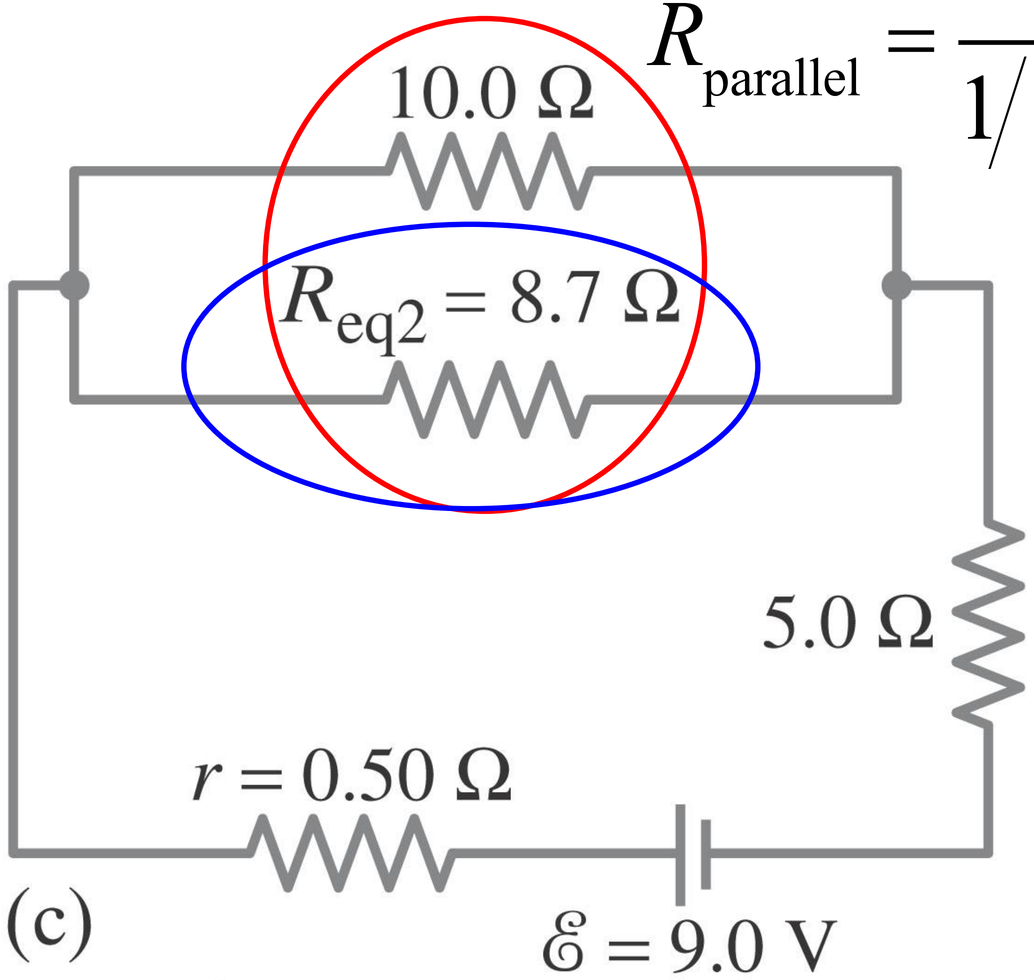
$$R_{\text{series}} = R_1 + R_2 \,.$$



Resistors in Series and in Parallel

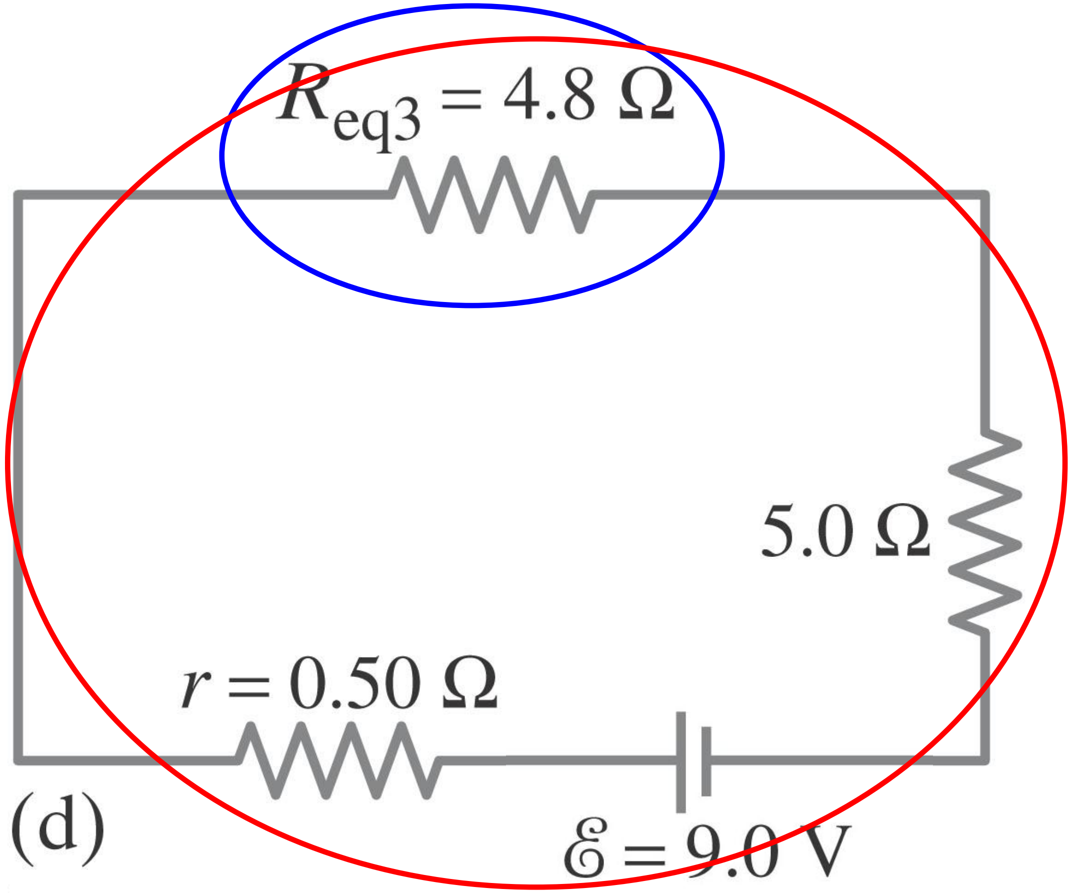
Example: Big circuit.

$$R_{\text{parallel}} = \frac{1}{1/R_1 + 1/R_2}$$



Resistors in Series and in Parallel

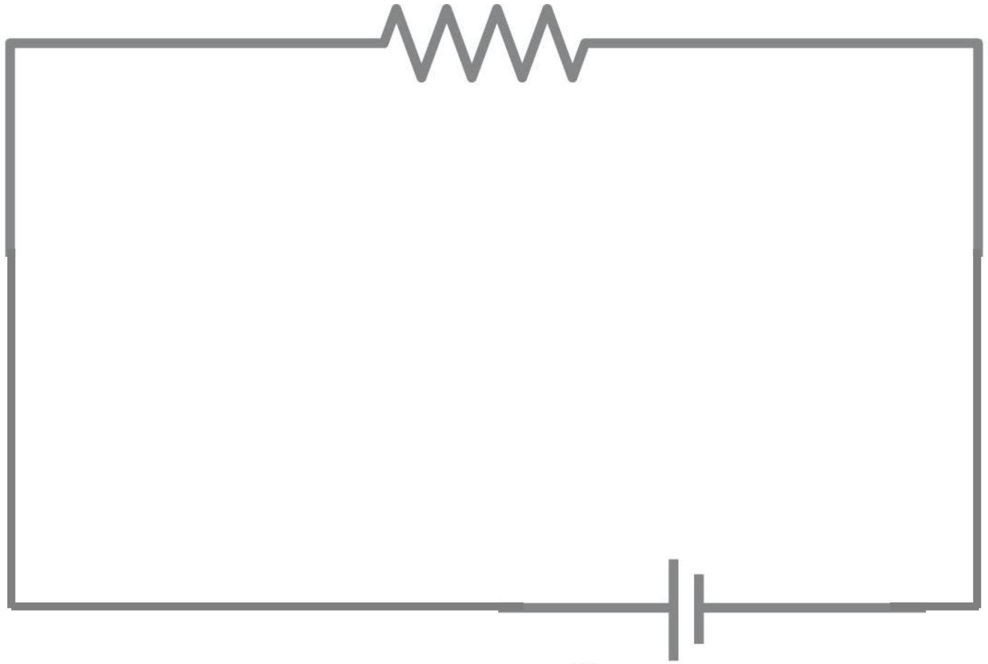
$$R_{\text{series}} = R_1 + R_2 + R_3$$



Resistors in Series and in Parallel

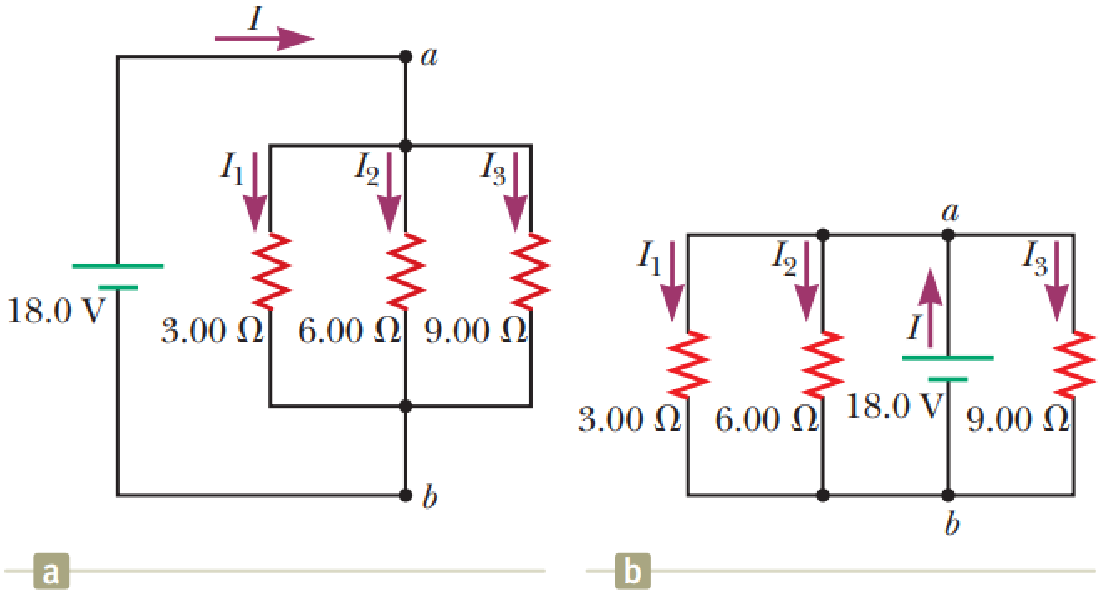
Example: Big circuit.

$$I = \frac{\mathcal{E}}{R_{tot}} = 0.873\ \text{A}$$



Example

Three resistors are connected in parallel as shown in Figure 28.11a. A potential difference of 18.0 V is maintained between points a and b.



(A) Calculate the equivalent resistance of the circuit.

Because the three resistors are connected in parallel, we can use the rule for resistors in parallel, to evaluate the equivalent resistance.

$$\frac{1}{R_{eq}} = \frac{1}{3.00\ \Omega} + \frac{1}{6.00\ \Omega} + \frac{1}{9.00\ \Omega} = \frac{11}{18.0\ \Omega}$$

$$R_{eq} = \frac{18.0\ \Omega}{11} = 1.64\ \Omega$$

(B) Find the current in each resistor

The potential difference across each resistor is 18.0 V. Apply the relationship ΔV = IR to find the currents:

$$I_1 = \frac{\Delta V}{R_1} = \frac{18.0\ \text{V}}{3.00\ \Omega} = 6.00\ \text{A}$$

$$I_2 = \frac{\Delta V}{R_2} = \frac{18.0\ \text{V}}{6.00\ \Omega} = 3.00\ \text{A}$$

$$I_3 = \frac{\Delta V}{R_3} = \frac{18.0\ \text{V}}{9.00\ \Omega} = 2.00\ \text{A}$$

(C) Calculate the power delivered to each resistor and the total power delivered to the combination of resistors

$$3.00\text{-}\Omega\text{: } P_1 = I_1^2 R_1 = (6.00\ \text{A})^2(3.00\ \Omega) = 108\ \text{W}$$

$$6.00\text{-}\Omega\text{: } P_2 = I_2^2 R_2 = (3.00\ \text{A})^2(6.00\ \Omega) = 54\ \text{W}$$

$$9.00\text{-}\Omega\text{: } P_3 = I_3^2 R_3 = (2.00\ \text{A})^2(9.00\ \Omega) = 36\ \text{W}$$

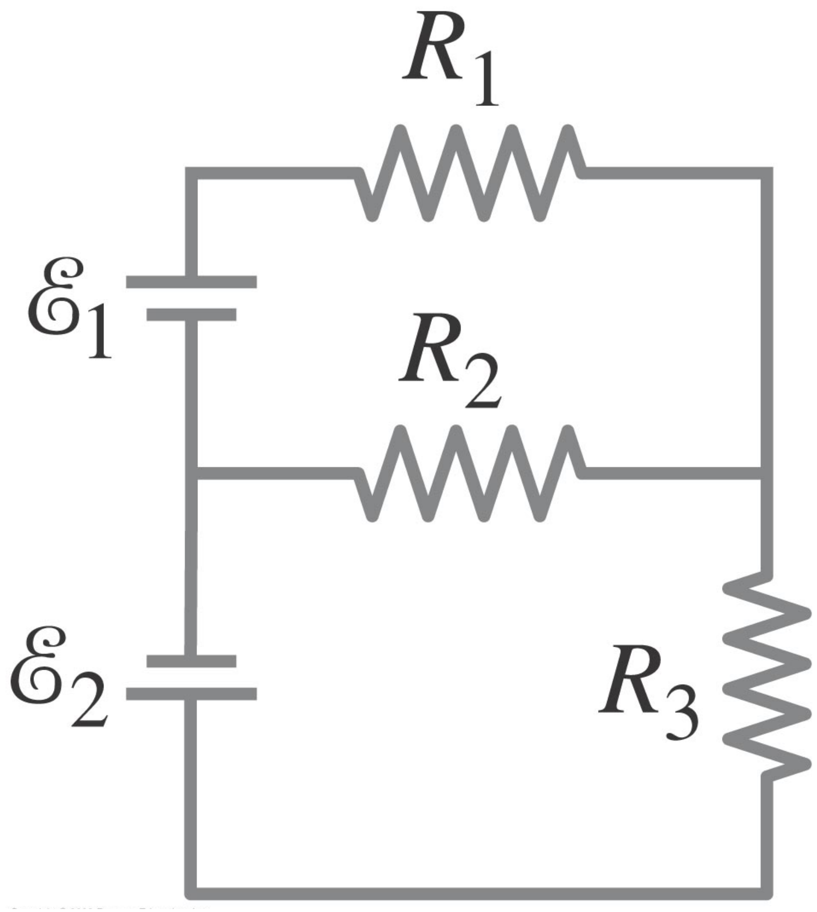
Summing the three quantities gives a total power of 198 W

$$P = (\Delta V)^2/R_{eq} = (18.0\ \text{V})^2/1.64\ \Omega = 198\ \text{W}.$$

Kirchhoff's Rules

Some circuits cannot be broken down into series and parallel connections (i.e. not reducible).

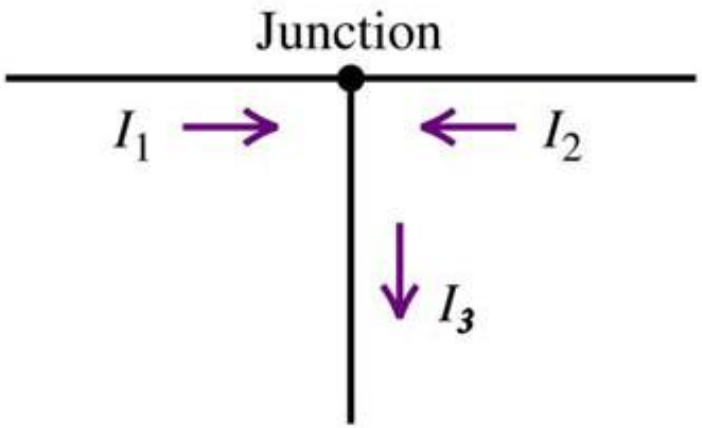
There is no way to reduce the three resistors to one effective resistance or to combine the two voltage sources to one voltage source.



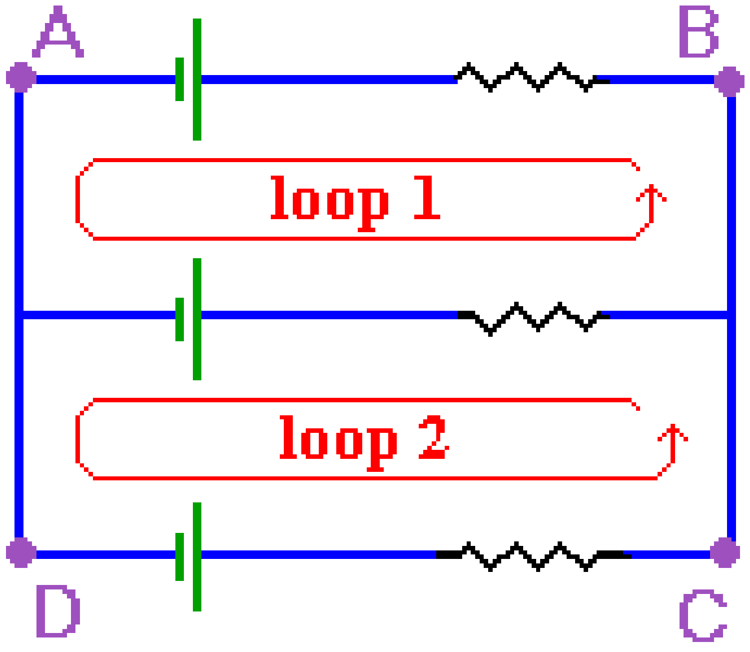
For such circuits we use Kirchhoff's rules.

Kirchhoff's Rules

A ***junction***, also called a ***node*** or ***branch point***, is a point where three or more conductors meet.



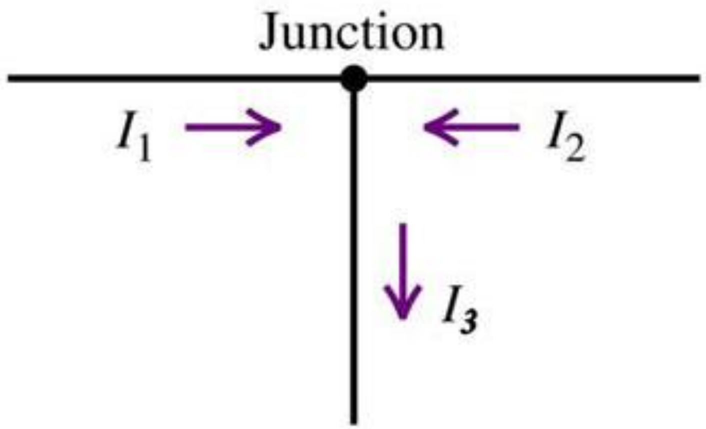
A ***loop*** is any closed conducting path.



Kirchhoff's Rules

Junction rule: The sum of the currents entering a junction is equal to the sum of leaving currents.

Kirchhoff's first rule is a statement of conservation of electric charge. All charges that enter a given point in a circuit must leave that point because charge cannot build up or disappear at a point.



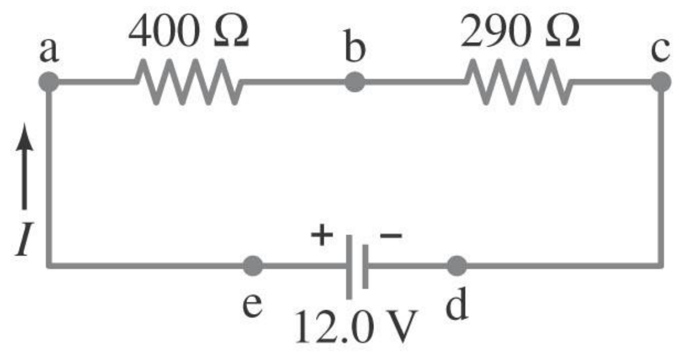
$$I_1 + I_2 = I_3 \qquad \sum I_{in} = \sum I_{out}$$

Kirchhoff's Rules

Loop rule: The algebraic sum of the emfs (batteries) in a loop

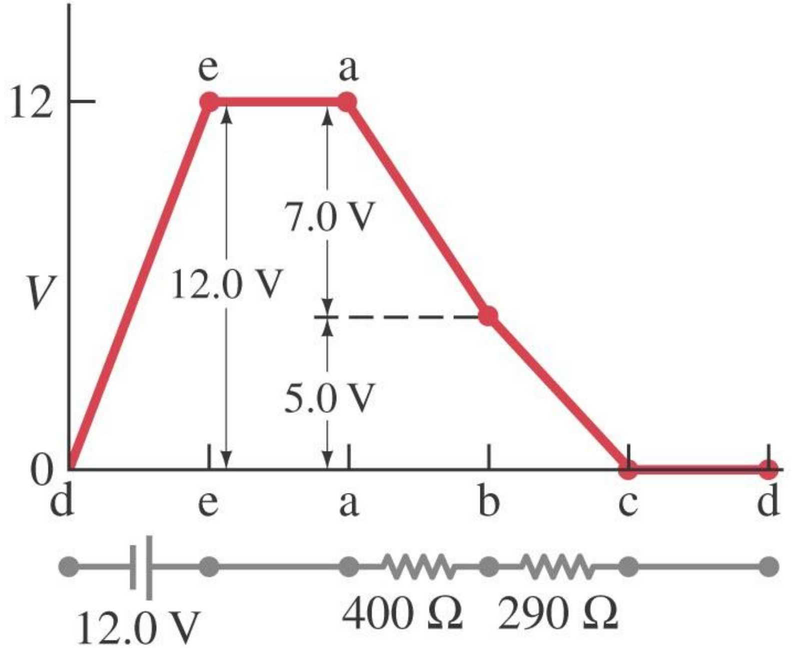
is equal

to the sum of potentials drops on all other (inactive) elements in the loop.



(a)

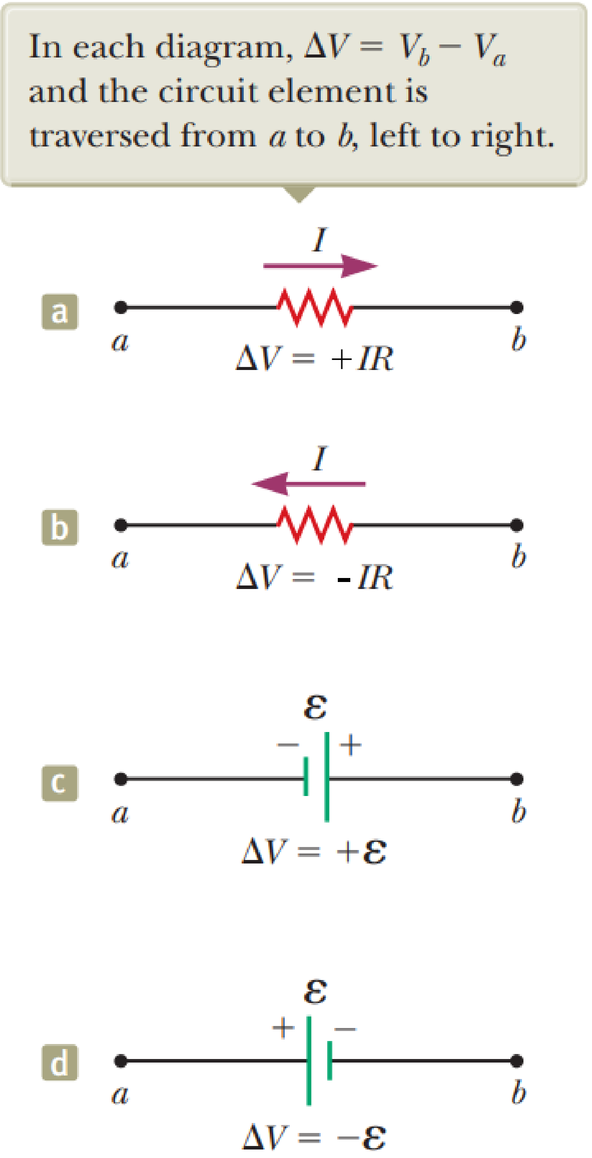
Kirchhoff's second rule follows from the law of conservation of energy for an isolated system. Let's imagine moving a charge around a closed loop of a circuit. When the charge returns to the starting point, the charge–circuit system must have the same total energy as it had before the charge was moved.



$$\sum_{batteries} \varepsilon_k = \sum_{R,C,L} V_{ij}$$

$$V_{ac}=V_{ab}+V_{bc}=V_{ae}+V_{ed}+V_{dc}$$

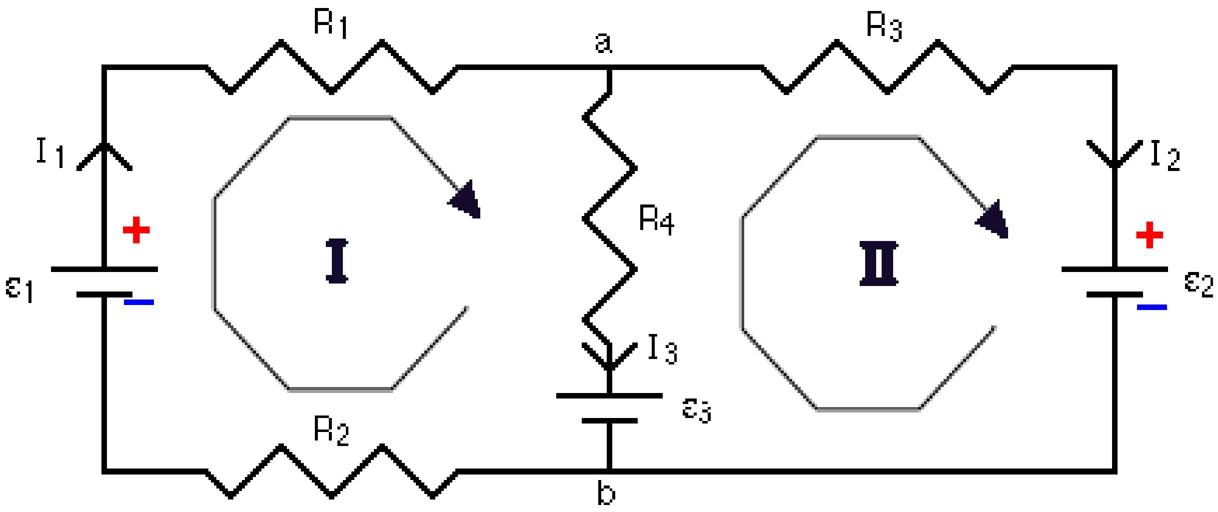
Kirchhoff's Rules



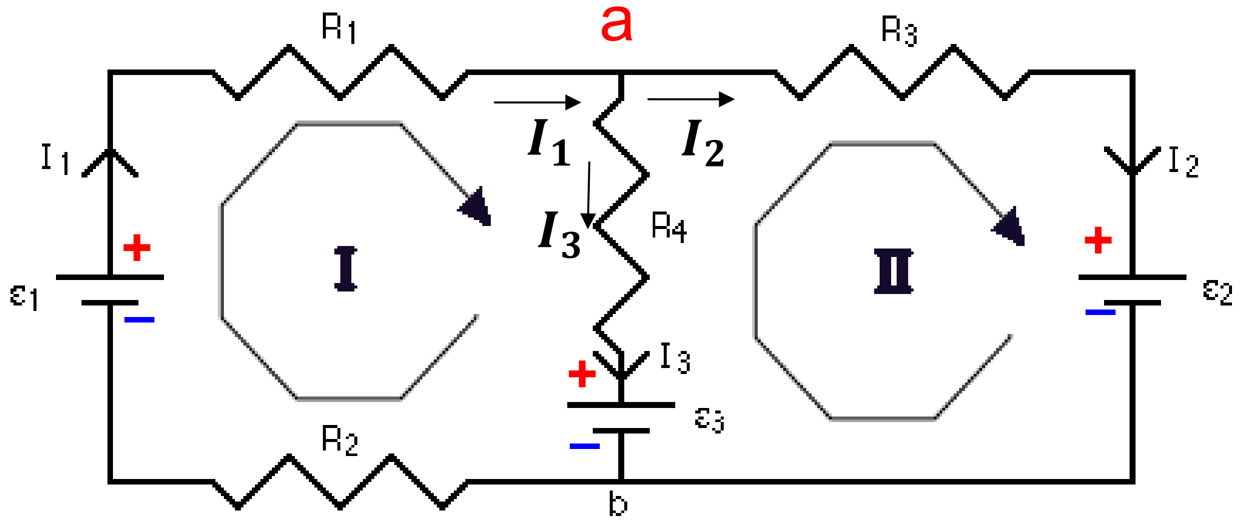
- **Choose the loops and the directions of currents.**

In the formulation $\sum_{batteries} \varepsilon_k = \sum_{R,C,L} V_{ij}$ **:**

- **A $\Delta V = emf$ term is considered to be positive if we go in the direction − to +, otherwise it is negative.**
- **A $\Delta V = IR$ term is positive if we cross R in the same sense as the current that is going through it, otherwise it is negative.**



Kirchhoff's Rules



For loop I we have $$\mathcal{E}_1 - \mathcal{E}_3 = I_1R_1 + I_3R_4 + I_1R_2$$

For loop II we have $$\mathcal{E}_3 - \mathcal{E}_2 = I_2R_3 - I_3R_4$$

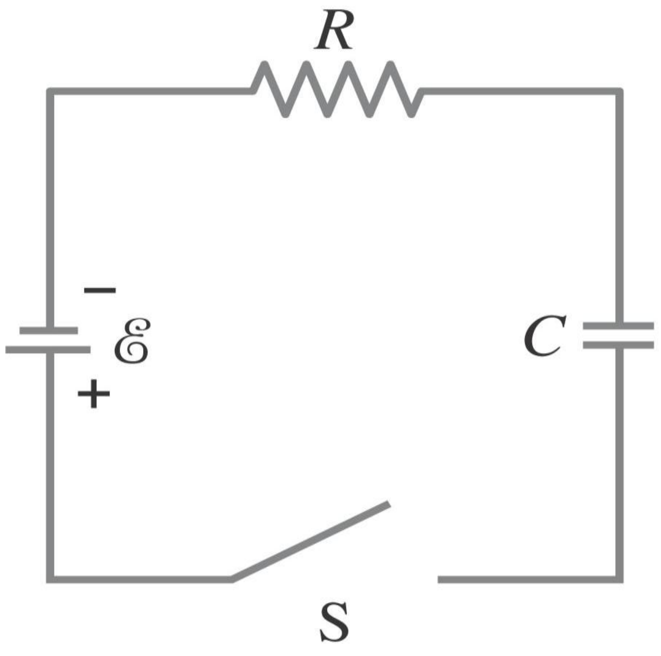
Junction equation at *a* gives us $$I_1 = I_2 + I_3$$

We now have three equations for the three unknown currents.

Circuits Containing Resistor and Capacitor (RC Circuits)

When the switch is closed, the capacitor will begin to charge. As it does, the voltage across it increases, and the current through the resistor decreases.

- Empty capacitor behaves like R=0
- Charged capacitor is like R= ∞
- The $\boldsymbol{\Delta V(t)}$ at the extremities of a capacitor is $Q(\boldsymbol{t})/\boldsymbol{C}$



How the charge, potential and current depend on time ?

Circuits Containing Resistor and Capacitor (RC Circuits)

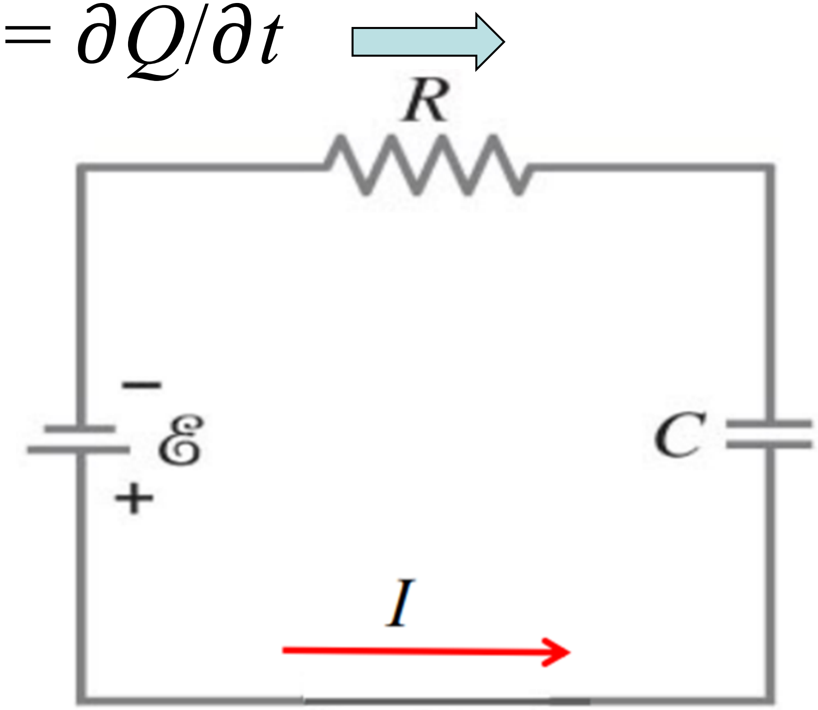
Kirchhoff' equation for the voltage around the loop:

$$\mathcal{E} = IR + \frac{Q}{C}$$

- *This equation is valid at any time t*

$$\mathcal{E} = I(t)R + \frac{Q(t)}{C}$$

Recall: $I = \partial Q/\partial t$



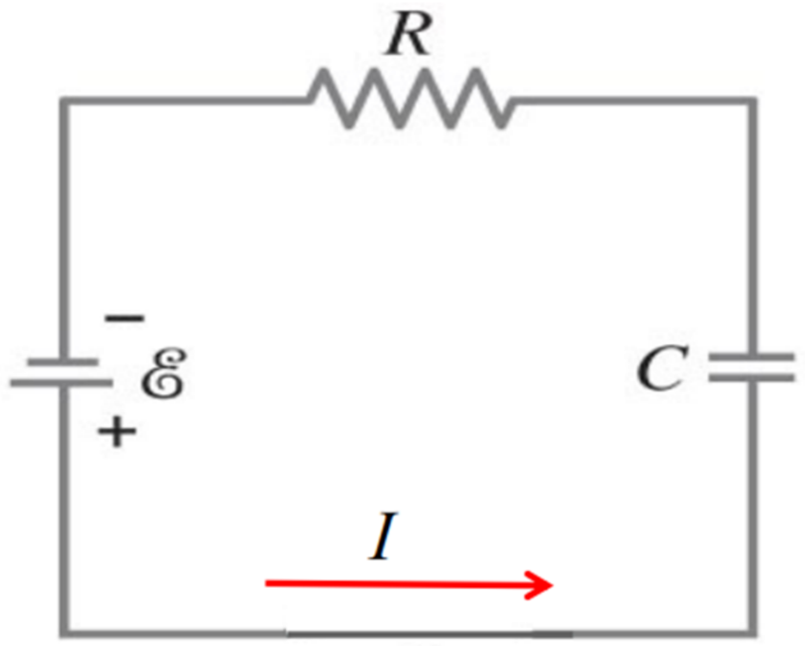
Time-dependent equation:

$$\mathcal{E} = \frac{\partial Q(t)}{\partial t}R + \frac{Q(t)}{C}$$

Let's solve it !

RC Circuits

$$\mathcal{E} = \frac{\partial Q(t)}{\partial t} R + \frac{Q(t)}{C}$$



Initial conditions:

at $\boldsymbol{t = 0}$ no charge ($\boldsymbol{Q = 0}$) on C

We can easily find the initial current $\boldsymbol{I_i}$ in the circuit and the maximum charge $\boldsymbol{Q_{max}}$ on the capacitor (for $\boldsymbol{t = \infty}$)

$I_i = \frac{\varepsilon}{R}$ (current at $t=0$)

$Q\,(t=0) = 0$

$Q_{max} = C\varepsilon$ (maximum charge)

$I\,(t=\infty) = 0$

RC Circuits

$$\varepsilon = \frac{\partial Q}{\partial t}R + \frac{Q}{C}$$

multiply by *C and rearrange*:

$$\frac{\partial Q}{\partial t}RC = \varepsilon C - Q$$

$$y = \varepsilon C - Q$$

differentiate:

$$\partial y = \partial(\varepsilon C - Q) = -\partial Q$$

Integrate (separation of variable)

$$-\frac{\partial y}{\partial t} = y\frac{1}{RC} \Rightarrow \frac{\partial y}{y} = -\frac{\partial t}{RC} \Rightarrow \ln(y) = -\frac{t}{RC} + const$$

Put to exp to both sides: $y = Aexp(-\frac{t}{RC})$

Replace y back: $\varepsilon C - Q = Aexp(-\frac{t}{RC})$

Initial conditions: $for\ t = 0$, $Q = 0$ $\Leftrightarrow$ $A = \varepsilon C$

$$Q = \varepsilon C\left[1 - \exp\left(-\frac{t}{RC}\right)\right]$$

RC Circuits: CHARGE

$$Q = \mathcal{E}C \cdot (1 - \exp(-\frac{t}{RC}))$$

$$Q_f = \lim_{t\to\infty} Q = \lim_{t\to\infty} C\mathcal{E}\left(1 - e^{-t/RC}\right) = C\mathcal{E}$$

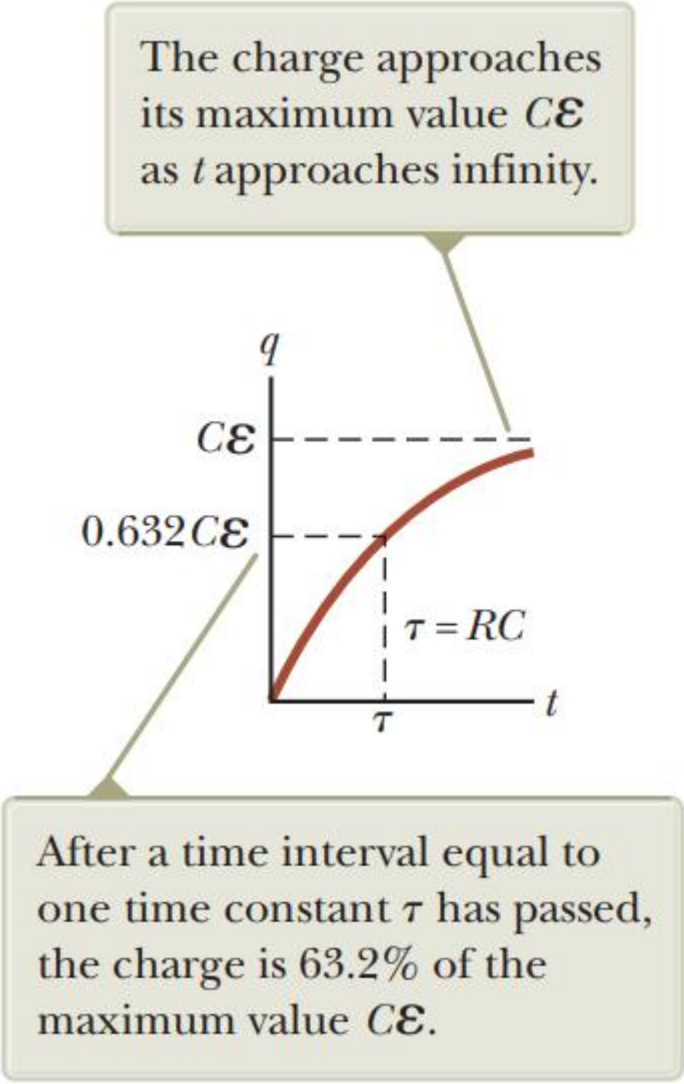
$$Q = Q_f \cdot (1 - \exp(-\frac{t}{RC}))$$

The quantity *RC* that appears in the exponent is called the time constant of the circuit:

$$\tau = RC.$$

- After this time Q=(1-1/e)= 63% of its final value

$$[\tau] = [RC] = \left[\left(\frac{\Delta V}{I}\right)\left(\frac{Q}{\Delta V}\right)\right] = \left[\frac{Q}{Q/\Delta t}\right] = [\Delta t] = \mathrm{T}$$



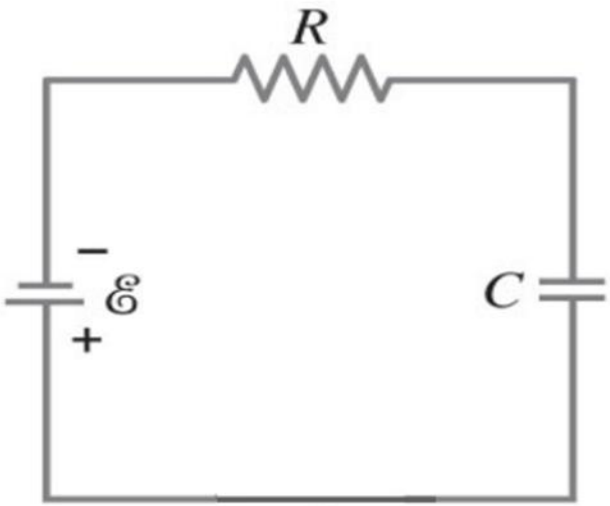
RC Circuits: VOLTAGE

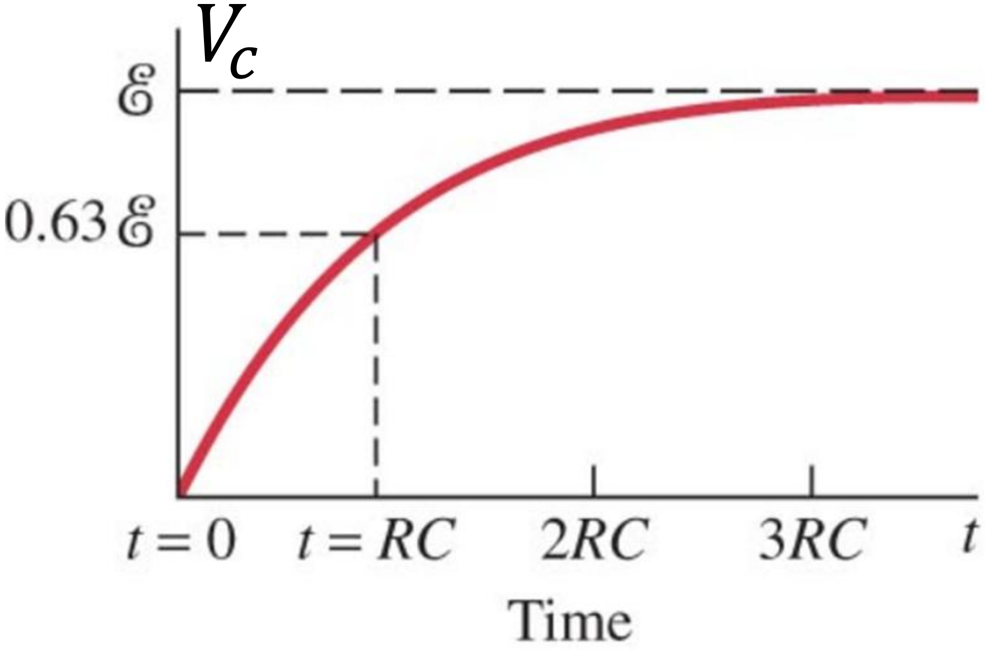
$$Q = \mathcal{E}C\cdot(1-\exp(-\frac{t}{RC}))$$

The voltage across the capacitor is $V_C = Q/C$:

$$V_C = \mathcal{E}\left(1-e^{-t/RC}\right)$$

- *Voltage on capacitor follows the charge*





RC Circuits: CURRENT

$$Q = \mathcal{E}C\cdot(1-\exp(-\frac{t}{RC}))$$

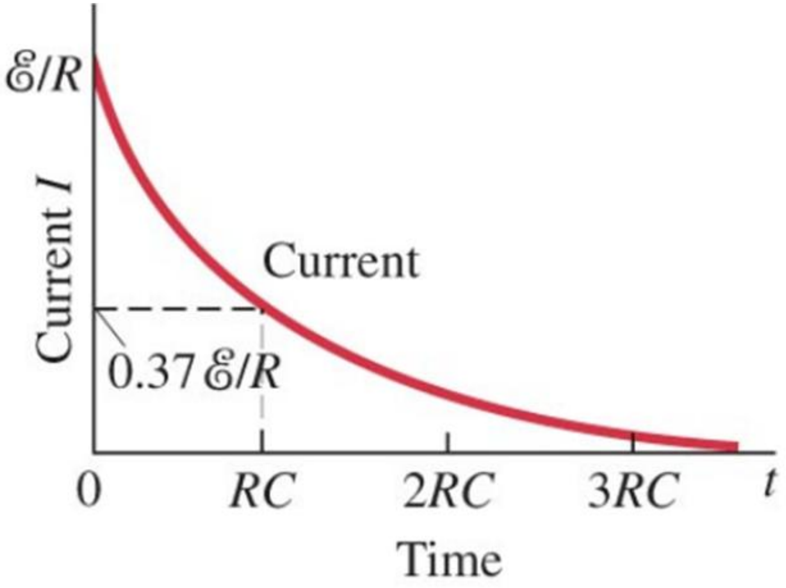
Differentiate the charge:

$$I = \frac{dQ}{dt} = \frac{1}{-RC}\frac{dQ}{d(-t/RC)}$$

$$y = -\frac{t}{RC}$$

$$I = \frac{dQ}{dt} = \frac{d[\varepsilon C(1-\exp(y))]}{dy} * \frac{dy}{dt}$$

$$I = \frac{dQ}{dt} = -\varepsilon C\exp(y) * (-\frac{1}{RC})$$



Current decays with time constant RC

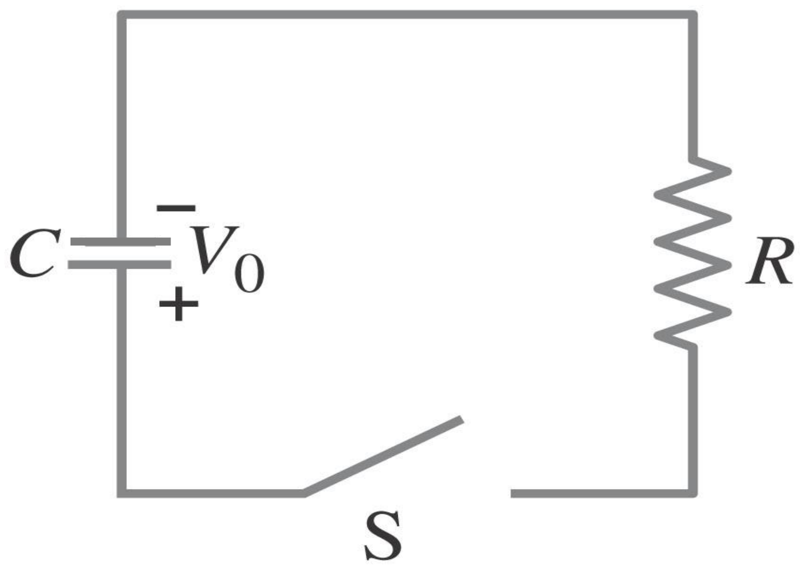
$$I = \frac{\mathcal{E}}{R} e^{-t/RC}$$

RC Circuits: discharging

Circuit: charged capacitor in series with a resistor and an open switch (starting condition).

The capacitor will now act as a source of emf (but not constant in time, as for a battery).

The capacitor will discharge and its voltage and emf will decrease in time.



Let's solve it !

RC Circuits: discharging

Kirchhoff's 2nd rule gives us

$$\frac{Q}{C} = IR.$$

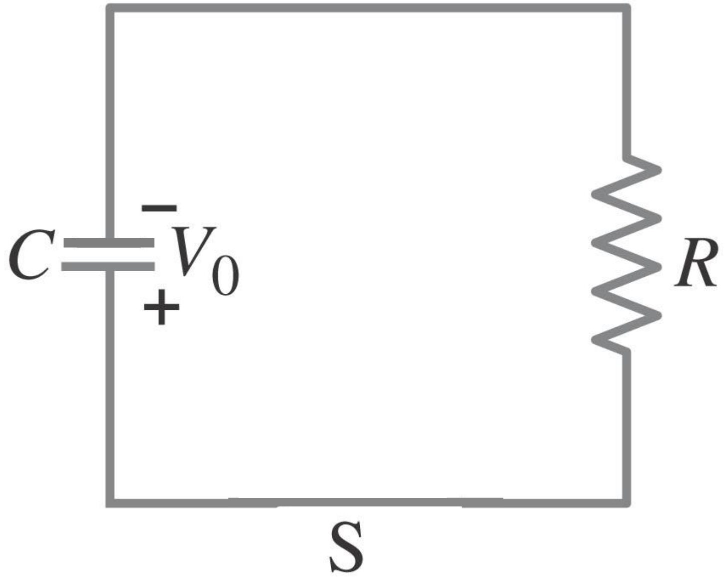
As the capacitor discharges

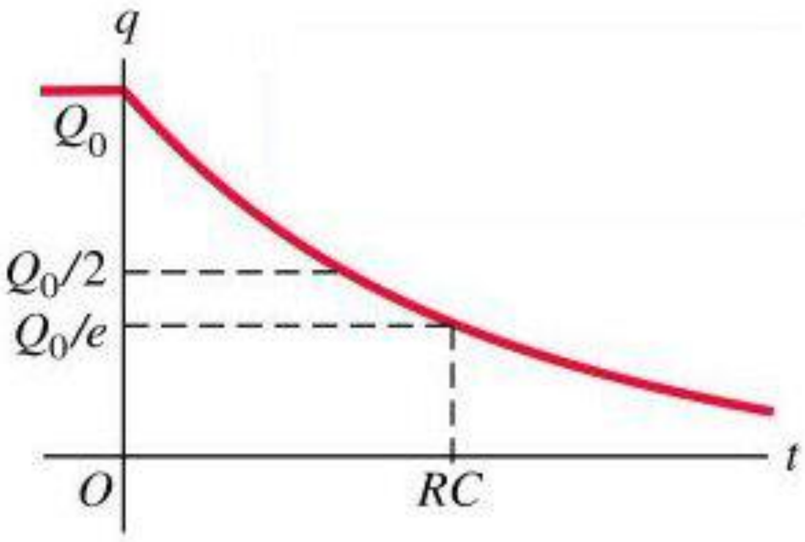
$$I = -\frac{dQ}{dt}.$$

Therefore,

$$\frac{Q}{C} = -\frac{dQ}{dt}R \rightarrow Q = Q_0 e^{-t/RC}.$$

- 1/e of Q_0 will be left at ***t*** =***R***C





RC Circuits: discharging

$$Q = Q_0 e^{-t/RC}.$$

Once again, the voltage and current as a function of time can be found from the charge:

$$V_C = \frac{Q}{C} = \frac{Q_0}{C} e^{-t/RC} = V_0 e^{-t/RC}$$

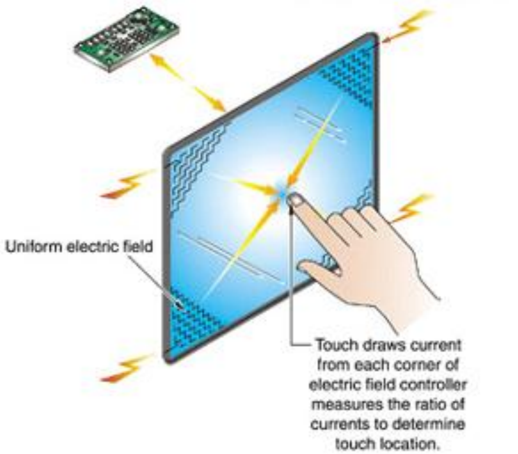
and

$$I = -\frac{dQ}{dt} \rightarrow I = \frac{Q_0}{RC} e^{-t/RC} = I_0 e^{-t/RC}.$$

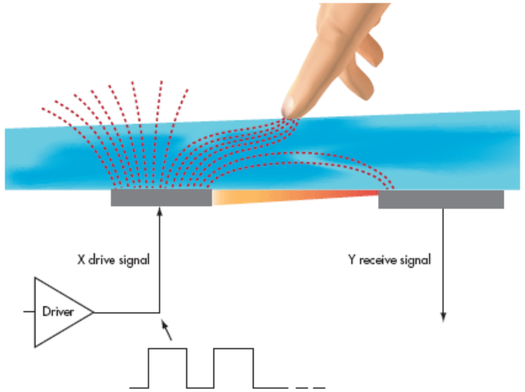
- Q, V_C and I_C decay with time constant τ = RC
- For a small R the discharge can be short but with high current !

Electrostatics and Microtechnology

Touchscreen



3. One projected touchscreen technology involves sensing along both the X- and Y-axis using clearly etched ITO patterns.



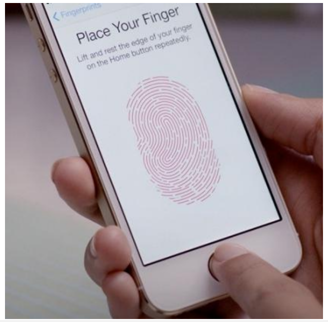
Piezoelectric "motors" for micro/nanometric positioning (with capacitive position sensors)

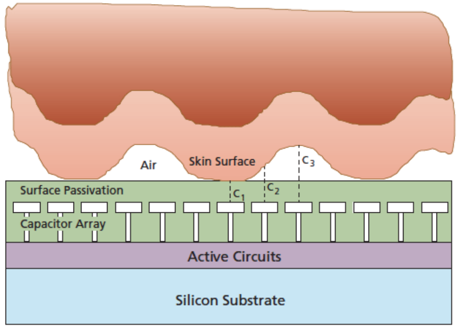
1-axis

2-axis

3-axis

Fingerprint sensor





Summary of electrostatics